DSF–49–2005
hep–th/0601056



# Twisted Conformal Symmetry in Noncommutative Two-Dimensional Quantum Field Theory

Fedele Lizzi*, Sachindeo Vaidya† and Patrizia Vitale*

*Dipartimento di Scienze Fisiche,
Università di Napoli Federico II and INFN Sezione di Napoli,
Via Cintia, 80126, Napoli, ITALY.

†Centre for High Energy Physics,
Indian Institute of Science,
Bangalore, INDIA 560012.

(Dated: February 1, 2008)

## Abstract

By twisting the commutation relations between creation and annihilation operators, we show that quantum conformal invariance can be implemented in the 2-d Moyal plane. This is an explicit realization of an infinite dimensional symmetry as a quantum algebra.

PACS numbers: 11.10.Nx,11.25.Hf

*Dedicated to Rafael Sorkin on the occasion of his 60th birthday.*

Conformal theories on the two-dimensional plane [1] play an important role in several aspects of modern physics, from string theory to applications to condensed matter. In this article, we will investigate the fate of this infinite dimensional symmetry when the plane is deformed to become the simplest *noncommutative geometry*. We will consider a plane in which the coordinates do not commute $[x_0, x_1]_\star = i\theta$, where $\theta$ is a constant with the dimensions of the square of length (in the units $c = 1$), and the meaning of the $\star$ subscript will be clear in the following. We will demonstrate that it is possible to construct a *quantum* field theory that is *quantum* both in the sense that fields are operators on a Fock space, and that the symmetry is realized as a quantum algebra. This provides an explicit example of the common believe that quantum algebras describe the symmetries of noncommutative geometries. An alternative approach to conformal symmetry on the 2-d Moyal plane which has points of overlap with this work is due to A. P. Balachandran, A. Marques and P. Teotonio-Sobrinho (to appear).

It is sometimes argued that the presence of an intrinsic length scale is incompatible with conformal invariance. This is no more correct than saying that the existence of a fundamental constant with dimensions of angular momentum necessarily destroys rotational invariance.The point is the manner in which the symmetry is implemented on the space of interest.

Consider fields on the noncommutative plane $\mathbb{R}^2_\theta$. Functions on this space are multiplied with the Moyal star product

$$(f \star g)(x) = e^{\frac{i}{2}\theta(\partial_{x_0}\partial_{y_1} - \partial_{x_1}\partial_{y_0})} f(x) \cdot g(y)|_{x=y} \tag{1}$$

For our purposes it is useful to see this product written as an ordinary product $m_0$ of a "twisted" tensor product:

$$(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g] \tag{2}$$

where $m_0(f \otimes g) = f \cdot g$ is the ordinary product and

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\otimes\partial_{y_\nu}} = e^{-\frac{i}{2}\theta(\partial_{x_0}\otimes\partial_{y_1} - \partial_{x_1}\otimes\partial_{y_0})} \tag{3}$$

is the *twist*.

Our starting point is the observation in [2, 3, 4] that when a product is a twist deformation of the commutative product, infinitesimal symmetries that act on the space are

implemented by deforming the coproduct by the same twist. In [5, 6] a global approach is undertaken, where the infinitesimal symmetries, together with their deformed coproduct, are seen to descend from a deformation of the product on the group manifold. The original study concerned the Poincaré symmetry on $\mathbb{R}^{1,3}_\theta$, and there has been subsequent work for conformal symmetries [7] in dimensions greater than 2, as well as diffeomorphisms [8] and gauge transformations [9].

To understand how infinitesimal generators of a given Lie algebra $\mathcal{G}$ act on the noncommutative plane, let us have a new look at their action in the commutative case. Given an infinitesimal generator $X$, we have

$$X(f \cdot g) = Xf \cdot g + f \cdot Xg \tag{4}$$

which we can rewrite as

$$X(f \cdot g) = m_0[(\mathbb{I} \otimes X + X \otimes \mathbb{I})(f \otimes g)] \tag{5}$$

where

$$\mathbb{I} \otimes X + X \otimes \mathbb{I} \equiv \Delta_0 \tag{6}$$

is the standard coproduct in the enveloping algebra $U(\mathcal{G})$. It captures the ordinary Leibniz rule. However with the $\star$ product the rule is violated, infinitesimal generators are not anymore derivations of the new algebra:

$$X(f \star g) \neq Xf \star g + f \star Xg \tag{7}$$

This means that the coproduct in the enveloping algebra has to be changed, hence $U(\mathcal{G}) \to U_\theta(\mathcal{G})$. Indeed we have

$$\begin{aligned} X(f \star g) &= X \cdot m_\theta(f \otimes g) = X \cdot m_0\big(\mathcal{F}^{-1} \cdot (f \otimes g)\big) \\ &= m_0\big(\Delta(X)\mathcal{F}^{-1} \cdot (f \otimes g)\big) \\ &= m_0\big(\mathcal{F}^{-1}\Delta_\theta(X) \cdot (f \otimes g)\big) \\ &= m_\theta\big(\Delta_\theta(X) \cdot (f \otimes g)\big) \end{aligned} \tag{8}$$

where the new coproduct is

$$\Delta_\theta(X) = \mathcal{F}^{-1}\Delta_0(X)\mathcal{F}. \tag{9}$$

The Lie algebra itself doesn't have to be changed. Namely, the action on single functions is *undeformed*. Notice, as a technical remark, that the deformed enveloping algebra $U_\theta(\mathcal{G})$, is a noncommutative, noncocommutative Hopf algebra, i.e. a *Quantum Group*.

Therefore, symmetries of noncommutative theories should be investigated in the form (8). A noncommutative theory will have a quantum symmetry if it is left invariant under the infinitesimal action (8). The authors above cited dealt with the classical symmetries of the Moyal space, while the extension of twisted symmetry to the quantum domain was done in [10]. To implement quantum symmetries we will take the point of view that whenever the tensor product of two fields appears, one always applies the twist. This means that all commutator among fields will automatically be $\star$-commutators, the representations will be combined with deformed coproducts, and that two-point correlation functions are calculated twisting the product of fields.

We want to investigate the fate of conformal symmetry in two space-time dimensions, when space-time is made noncommutative. To this, let us consider the two dimensional Minkowski plane in light cone coordinates. A treatment using complex coordinates is probably possible, and definitely interesting, but we will not do it here. We will use the convention in which the spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1)$. Light cone coordinates are defined as $x^\pm = x^0 \pm x^1$. Using $\eta_{AB}dx^A dx^B = \eta_{\mu\nu}dx^\mu dx^\nu$, and $\eta^{AB} = (\eta_{AB})^{-1}$ (here $A, B$ label lightcone indices + and -) we have

$$\begin{aligned} \eta_{++} = \eta_{--} &= 0 = \eta^{++} = \eta^{--}, \\ \eta_{+-} = \eta_{-+} &= \tfrac{1}{2} = (\eta^{+-})^{-1} = (\eta^{-+})^{-1}. \end{aligned} \tag{10}$$

From $x_A = \eta_{AB}x^B$ we also have the rule $x_\pm = x^\mp/2$ .

Let us briefly recall the situation in the commutative case. In two dimensions the infinite dimensional conformal algebra is generated by vector fields $u^\mu(x)\partial_\mu$ that satisfy

$$\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu}\partial^\alpha u_\alpha = 0. \tag{11}$$

In light cone coordinates, this is $\partial_+ u_+ = 0 = \partial_- u_-$, which implies that

$$\begin{aligned} u_+ = u_+(x^+) &\Leftrightarrow u^- = u^-(x_-) \quad \text{and} \\ u_- = u_-(x^-) &\Leftrightarrow u^+ = u^+(x_+) \end{aligned} \tag{12}$$

The + and $-$ sectors are independent under conformal transformations. Considering $u_\pm$ as being generated by $-(x^\pm)^{n+1}$, we have that conformal transformations are in turn generated

by the vector fields

$$\ell_n^{\pm} = -(x^{\pm})^{n+1}\partial_{\pm}, \quad \text{with} \quad n \in \mathbb{Z}. \tag{13}$$

They form two copies of the classical Virasoro (Witt) algebra

$$\begin{aligned} [\ell_n^{\pm}, \ell_m^{\pm}] &= (m-n)\ell_{m+n}^{\pm} \\ [\ell_n^{+}, \ell_m^{-}] &= 0 \end{aligned} \tag{14}$$

On the noncommutative plane we still implement conformal transformations via the $\ell_n$'s. The twist (3) however causes a change in the coalgebra structure. Using (9) we have

$$\begin{aligned} \Delta_\theta(\ell_n^{+}) &= (\mathbf{1}\otimes x^{+} - \theta\partial_{-}\otimes\mathbf{1})^{n+1}(-\mathbf{1}\otimes\partial_{+}) + (x^{+}\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_{-})^{n+1}(-\partial_{+}\otimes\mathbf{1}) \\ \Delta_\theta(\ell_n^{-}) &= (-\mathbf{1}\otimes\partial_{-})(\mathbf{1}\otimes x^{-} - \theta\partial_{+}\otimes\mathbf{1})^{n+1} + (-\partial_{-}\otimes\mathbf{1})(x^{-}\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_{+})^{n+1}. \end{aligned} \tag{15}$$

(These relations have also been independently derived by S. Kurkcuoglu [11].) The fact that the commutation relations of the Lie algebra are unchanged, and that $\theta$ is only present in the coproduct means that the effect of the twist is only relevant when fields are combined, for example in correlation functions.

The above basis, while convenient for deriving the commutation relations between generators, is inappropriate to study the invariance of quantum theories because it is not normalizable. Our derivation of the twisted version of the quantum Virasoro algebra will be in a different guise. To this end, consider the simplest two dimensional conformally invariant theory, a scalar massless field theory. On the noncommutative plane we have to use the star product:

$$S = \int d^2x \; \partial_{+}\varphi \star \partial_{-}\varphi \; . \tag{16}$$

The classical noncommutative action can be seen to be twist-conformally invariant, along the lines of [3], provided one uses the appropriate twist of the generators (15). The classical solutions are fields split in "left" and 'right" movers $\varphi = \varphi_{+}(x^{+}) + \varphi_{-}(x^{-})$. Although this theory is free (and the star product would drop from the integral), it is still necessary to check its invariance under the action of the conformal coalgebra. An analysis of other theories and the behaviour of the S-matrix, including the UV/IR mixing (or lack of it) will be presented in a longer paper in preparation.

In the quantum theory the fields are operators with mode expansion

$$\phi(x^0, x^1) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k_0} \left( a(k) e^{-ik^\mu x_\mu} + a^\dagger(k) e^{ik^\mu x_\mu} \right). \tag{17}$$

Using $k^0 = k_0 = |k^1|$ this can be rewritten as

$$\phi(x^+, x^-) = \int_{-\infty}^{0} \frac{dk^1}{4\pi|k^1|} \left( a(k) e^{-i|k^1|x^+} + a^\dagger(k) e^{i|k^1|x^+} \right) + \int_{0}^{\infty} \frac{dk^1}{4\pi|k^1|} \left( a(k) e^{-ik^1 x^-} + a^\dagger(k) e^{ik^1 x^-} \right). \tag{18}$$

This in turn may be rewritten as

$$\phi(x^+, x^-) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi|k^1|} \left( a_-(k) e^{-ik^1 x^+} + a_+(k) e^{-ik^1 x^-} \right) \tag{19}$$

where we have introduced the symbol $a_\sigma(k),\ \sigma = +, -,\ k \in (-\infty, \infty)$ related to the two sets of oscillators appearing in (18) (left and right movers) by $a_\sigma(k) = a(\sigma k),\ a_\sigma(-k) = a^\dagger(-\sigma k),\ k \in (0, \infty)$ .

In the standard case the commutation relations for the creation and annihilation operators are

$$[a_\sigma(p), a_{\sigma'}(q)] = 2p\delta(p+q)\delta_{\sigma\sigma'}. \tag{20}$$

Then the quantum currents $J^+(x) = \partial^+\phi,\ \ J^-(x) = \partial^-\phi$ generate two commuting U(1) Kaç-Moody algebras with central extension. Quantum conformal invariance is proved showing that the components of the quantum stress-energy tensor

$$\begin{aligned} \Theta_{\pm\pm} &= \frac{1}{4}(\Theta_{00} \pm 2\Theta_{01} + \Theta_{11}), \\ \Theta_{+-} &= \frac{1}{4}(\Theta_{00} - \Theta_{11}) = \Theta_{-+} \end{aligned} \tag{21}$$

generate the conformal algebra. Tracelessness and conservation ($\partial^\mu\Theta_{\mu\nu} = 0$) imply $\Theta_{\pm\mp} = 0$ and $\partial^\pm\Theta_{\pm\pm} = 0$. Hence $\Theta_{\pm\pm}(= \frac{\Theta^{\mp\mp}}{4})$ is a function of $x^\mp$ only, as in the standard case. Classically, $\Theta^{++}(x) = J^+(x)J^+(x)$. But as is well-known, the quantum stress-energy tensor is the *normal-ordered* product

$$\Theta^{\pm\pm}(x) = \gamma : J^\pm(x) J^\pm(x) : , \tag{22}$$

where $\gamma$ is a real number which gets fixed in the quantum theory, and normal ordering is defined as

$$\begin{aligned} : a_\sigma(p) a_\sigma(q) : &= a_\sigma(p) a_\sigma(q) \text{ if } p < q \\ : a_\sigma(p) a_\sigma(q) : &= a_\sigma(q) a_\sigma(p) \text{ if } p \geq q. \end{aligned} \tag{23}$$

Therefore the existence of Kaç-Moody quantum current algebras is a sufficient condition to ensure conformal invariance at the quantum level. In the noncommutative case things do not carry through unchanged and some crucial adjustments have to be performed.

The Kaç-Moody currents are

$$\begin{aligned} J^- &= i\int_{-\infty}^{0} \frac{dk^1}{2\pi}\left(a^\dagger(k)e^{-ik^1x^+} - a(k)e^{ik^1x^+}\right) \\ J^+ &= i\int_{0}^{\infty} \frac{dk^1}{2\pi}\left(a^\dagger(k)e^{ik^1x^-} - a(k)e^{-ik^1x^-}\right) \end{aligned} \tag{24}$$

where the oscillators in $J^-, J^+$ are different, being connected to negative and positive frequencies respectively. We omit the spatial index for the momenta from now on.

These currents, *with the standard commutation relations* (20) do not give rise to the Kaç-Moody algebras. In fact there is no reason to expect the same commutation relations for the noncommutative plane.

To this end we pursue the following strategy for the quantum field theory on the Moyal plane. We require that the noncommutative Kaç-Moody relations be the same as the usual ones, but with $\star$-commutators instead of ordinary ones:

$$\begin{aligned} \left[J^\pm(x), J^\pm(y)\right]_\star &= -\frac{i}{\pi}\partial_\mp\delta(x^\mp - y^\mp), \\ \left[J^+(x), J^-(y)\right]_\star &= 0. \end{aligned} \tag{25}$$

This will yield a deformation in the commutation rules of creation and annihilation operators.

Let us consider the current commutators $[J^\pm(x), J^\pm(y)]_\star$, $[J^+(x), J^-(y)]_\star$. We combine fields at different points using the twist, so that, (with a slight abuse of notation)

$$f(x)\star f(y) = \left[\mathcal{F}(f\otimes g)\right](x,y) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}} f(x)f(y). \tag{26}$$

Our first observation is that the deformation in the product does not affect the commutators $[J^\pm(x), J^\pm(y)]_\star$ because in light-cone coordinates we have

$$e^{\frac{i}{2}\partial_x\wedge\partial_y} e^{\mp ikx^\pm} e^{\mp ipy^\pm} = e^{i\theta(\partial_{x^-}\partial_{y^+} - \partial_{x^+}\partial_{y^-})} e^{\mp ikx^\pm} e^{\mp ipy^\pm} = e^{\mp ikx^\pm} e^{\mp ipy^\pm} \tag{27}$$

with $\partial_x \wedge \partial_y = \theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}$. Hence in each chiral sector the symmetry is unchanged. This is to be expected since the $\star$-product between two functions of $x^+$ (or $x^-$) alone is the same as the usual product. Likewise the coproduct(15) when acting on pairs of such functions is the same as the undeformed one. The effect of the noncommutativity of the plane is only

felt when $x^+$ and $x^-$ are put together. Consider the remaining commutator $[J_+(x), J_-(y)]_\star$. For this we have to use

$$e^{\frac{i}{2}\partial_x\wedge\partial_y}e^{-ikx^+}e^{ipy^-} = e^{-i\theta kp}e^{-ikx^+}e^{ipy^-} \ . \tag{28}$$

It is not difficult to see that the commutation relations (20) do not give two commuting currents, and the theory would not be conformally invariant in any sense. It is however possible to still obtain two mutually commuting algebras with the following *deformation of the commutation relations* (20):

$$a_\sigma(p)a_{\sigma'}(q) = \mathcal{F}^{-1}(p,q)a_{\sigma'}(q)a_\sigma(p) + 2p\delta(p+q)\delta_{\sigma\sigma'}, \tag{29}$$

where

$$\mathcal{F}^{-1}(p,q) = e^{-\frac{i}{2}p\wedge q} = e^{-i\theta(|p|q-|q|p)} \tag{30}$$

is the inverse of the twist (3) in momentum space. Notice that the action of $\mathcal{F}^{-1}(p,q)$ is always zero when considering the commutator between currents of the same chirality because $p$ and $q$ have the same sign. Summarizing, thanks to relations (29) left and right currents commute, while currents of the same chirality yield a central term, as in the standard theory. This result, whose main ingredient is Eq. (29), descends from the choice of the star-commutator in the form

$$[\phi(x),\phi(y)]_\star \equiv \phi(x)\star\phi(y) - \phi(y)\star\phi(x). \tag{31}$$

Indeed, (29) imply in particular that the star-commutator of the $\phi$ fields is the standard one. Other choices could be undertaken (see for example [12, 13]), all of which equally legitimate, in the absence of a guiding principle, which is for the moment still missing.

The appearance in (29) of a "quantum plane-like" deformation of the commutation between creation and annihilation operators in a quantum field theory on the noncommutative plane is not new. It has already appeared in higher dimensional theories [10] where it is postulated a relation of the kind $a(p)a(q) = G(p,q)a(q)a(p)$ to preserve Lorentz invariance. The use of the twisted coproduct (and the proper limit for $\theta \to 0$) then fixes the quantity $G$ to be the twist in dimensions greater than 2. In $1+1$ dimensions Lorentz invariance is not enough to fix the precise form of the deformation of the commutation relations.

Now that we have the Kaç-Moody currents, we demonstrate our initial conjecture, that is, we construct the stress-energy tensor in terms of the Kaç-Moody currents and verify that its components generate the standard quantum conformal algebra.

Using the relations (29) and the twist, one can see that the commutators $[\Theta^{\pm}(x), \Theta^{\pm}(y)]$ are unchanged from the usual case (see for example [14]), as the twist has no effect on pairs of functions of $x^+$ (or $x^-$) only:

$$[\Theta^{\pm}(x), \Theta^{\pm}(y)]_\star = \pm\frac{4i}{\pi}\Theta^{\pm\pm}(x)\partial_\mp\delta(x-y) - \frac{i}{6\pi^3}\partial_\mp^{'''}\delta(x-y) \ . \tag{32}$$

The mixed commutator instead implies a nontrivial combination of (28), (29). A straightforward, although tedious, calculation gives

$$[\Theta^{++}(x), \Theta^{--}(y)]_\star = 0 \ , \tag{33}$$

thus confirming the existence of twisted conformal symmetry.

In this article we have shown that it is possible to build a 2-dimensional quantum field theory on the noncommutative plane which still possesses an infinite dimensional symmetry. This symmetry is a quantum deformation of the standard conformal algebra with undeformed Lie brackets. In [15] and [16] an interpretation has been suggested of twisted symmetries. While [16] argues that these are not standard physical symmetries because they modify the star product, in [15], upon performing an analysis of covariance of noncommutative field theories, it is shown that, while covariance amounts to invariance under "observer" dependent transformations, twist symmetries are "particle" dependent transformations, namely, transformations which modify localized fields but not background fields as the deformation tensor $\theta$ (for details on "observer" and "particle" dependent transformations we refer to [17]).

The task would now be to use this conformal theory to unveil in the noncommutative case as well, all the richness of this infinite quantum symmetry, exploring for example possible novel relations among vertex operators which come from the twist. We plan to return on this aspect, as well as a more detailed description of the symmetry, in a longer report in the near future.

It is a pleasure to thank P. Aschieri, A. P. Balachandran, S. Kurkcuoglu and J. Lukierski for discussions and correspondence. This work has been supported in part by the *Progetto di Ricerca di Interesse Nazionale* Sintesi.

---

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, "Infinite Conformal Symmetry In Two-Dimensional Quantum Field Theory," Nucl. Phys. B **241** (1984) 333.

[2] J. Wess, "Deformed coordinate spaces: Derivatives," arXiv:hep-th/0408080.

[3] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, "On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT," Phys. Lett. B **604**, 98 (2004) [arXiv:hep-th/0408069].

[4] M. Chaichian, P. Presnajder and A. Tureanu, "New concept of relativistic invariance in NC space-time: Twisted Poincare symmetry and its implications," Phys. Rev. Lett. **94**, 151602 (2005) [arXiv:hep-th/0409096].

[5] R. Oeckl, "Untwisting noncommutative $R^d$ and the equivalence of quantum field theories", Nucl. Phys. **B581** (2000) 559. [arXiv:hep-th/0003018].

[6] C. Gonera, P. Kosinski, P. Maslanka and S. Giller, "Global symmetries of noncommutative space-time," Phys. Rev. D **72**, 067702 (2005) [arXiv:hep-th/0507054]; "Space-time symmetry of noncommutative field theory," Phys. Lett. B **622**, 192 (2005) [arXiv:hep-th/0504132].

[7] P. Matlock, "Non-commutative geometry and twisted conformal symmetry," Phys. Rev. D **71**, 126007 (2005) [arXiv:hep-th/0504084].

[8] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, "A gravity theory on noncommutative spaces," Class. Quant. Grav. **22**, 3511 (2005) [arXiv:hep-th/0504183].

[9] D.V. Vassilevich, "Twist to close," [arXiv:hep-th/0602185]

[10] A. P. Balachandran, G. Mangano, A. Pinzul and S. Vaidya, "Spin and statistics on the Groenwald-Moyal plane: Pauli-forbidden levels and transitions," arXiv:hep-th/0508002.

[11] S. Kurkcuoglu (private communication).

[12] Jong-Geon Bu, Hyeong-Chan Kim, Youngone Lee, Chang Hyon Vac, Jae Hyung Yee, "Noncommutative field theory from twisted Fock space," [arXiv:hep-th/0603251]

[13] J. Zahn, "Remarks on twisted noncommutative quantum field theory," Phys. Rev. D **73**, 105005 (2006) [arXiv:hep-th/0603231].

[14] S. Fubini, A. J. Hanson and R. Jackiw, "New approach to field theory," Phys. Rev. **D7** 1732 (1973).

[15] J.M. Gracia-Bondia, Fedele Lizzi, F.Ruiz Ruiz, Patrizia Vitale, "Noncommutative spacetime symmetries:twist versus covariance," arXiv:hep-th/0604206.

[16] Luis Alvarez-Gaume, Frank Meyer, Miguel A. Vazquez-Mozo, "Comments on noncommutative gravity," arXiv:hep-th/0605113

[17] D. Colladay and V. A. Kostelecky, "Lorentz-violating extension of the standard model," Phys.

Rev. D **58**, 116002 (1998) [arXiv:hep-ph/9809521].